10.

a)

Y has moment generating function mY (z) = $e^{-8z+32z2}$. What is the variance of Y ? What is P(Y ≤ −4)?

Solution:

Moment generating function of X~N(μ,σ^2) is $M_X(z) = e^{z\mu+\frac{1}{2}\sigma^2 z^2}$

Since $M_Y(z) = e^{-8z+32z^2} = e^{z(-8)+\frac{1}{2}(64)z^2}$ then Y~N(-8,8^2)

So the variance of Y is 8^2=64

Since Y~N(-8,8^2) then Z = (Y+8)/8 has a standard normal distribution

P(Y≤ −4) = P(Z ≤ ½) = 0.6915

Answer: Var(Y) = 64 and P(Y≤ −4) = 0.6915

b)Use the moment generating function to derive the expected value of a Poisson random variable with parameter λ = 1/3.

Solution:

The moment generating function of a Poisson random variable (X) with parameter

λ = 1/3 Is: $M_X(z) = e^{\frac{1}{3}(e^z-1)}$

Also we know that E(X) = $M'_X(0)$

Since $M_X(z) = e^{\frac{1}{3}(e^z-1)}$ then:

$$M'_X(z) = e^{\frac{1}{3}(e^z-1)}\left(\frac{1}{3}\right)e^z \rightarrow M'_X(0) = e^{\frac{1}{3}(e^0-1)}\left(\frac{1}{3}\right)e^0 = \frac{1}{3}$$

So we proved that E(X) = 1/3

Answer: E(X) = 1/3

c)Let Y ~ Uniform(α,β), α,β ∈ R and α β−Y}= 1/(1+y)

Clarify this question

9. (a)

The random variable X has moment generating function

mX (t) = exp (−5t + 12t^2) .
If Z = 1/5 (X − 2), what are E(Z) and Var(Z)?

Solution:

Moment generating function of X~N(μ,σ^2) is $M_X(t) = e^{t\mu+\frac{1}{2}\sigma^2 t^2}$

Since $M_X(t) = e^{-5t+12t^2} = e^{t(-5)+\frac{1}{2}(24)t^2}$ then X~N(-5,24)

So E(X) = -5 and Var(X) = 24

E(Z) = E(1/5(X-2)) = (1/5)E(X-2) = (1/5)(E(X)-E(2)) = (1/5)(-5-2) = -7/5 = -1.4

Var(Z) = Var(1/5(X-2)) = (1/25)Var(X-2) = (1/25)Var(X) = 24/25 = 0.96

Answer: E(Z) = -1.4 ,Var(Z) = 0.96

(b) Y is a random variable with moment generating function mY (t) = (1/(1 − 2t))^8 .
What are E(Y) and Var(Y)?

Solution:

Moment generating function of X~λ^2_k is $M_X(t) = \frac{1}{(1-2t)^{k/2}}$, E(X) =k , Var(X) = 2k

Since $M_Y(t) = \frac{1}{(1-2t)^{16/2}}$ then Y~λ^2_{16} , E(Y) = 16 , Var(Y) = 32

Answer: E(Y) = 16 , Var(Y) = 32

8.
Let Z1 and Z2 be independent normal random variables where Z1 ∼ N(6,3^2) and Z2 ∼ N(4,1^2). Define Y1 = Z1 −Z2 and Y2 = 2(Z1) +3(Z2). Find the means and variances of the random variables Y1 and Y2. [6]

Solution

We know that $E(Z_1) = 6$, $Var(Z_1) = 9$, $E(Z_2) =4$, $Var(Z_2) = 1$

$E(Y_1) = E(Z_1-Z_2) = E(Z_1)-E(Z_2) = 6-4 = 2$

Var(Y1) = $Var(Z_1-Z_2) = Var(Z_1)+Var(-Z_2) = Var(Z_1)+Var(Z_2) = 9+1 = 10$

$E(Y_2) = E(2Z_1+3Z_2) = 2E(Z_1)+3E(Z_2) = 2(6)+3(4) = 24$

Var(Y2) = $Var(2Z_1+3Z_2) = 4Var(Z_1)+9Var(Z_2) = 4(9)+9(1) = 45$

Answer: $E(Y_1) = 2$, $E(Y_2) = 24$, $Var(Y_1) = 10$, $Var(Y_2) = 45$

The random variable X has moment generating function mX (t) = exp(−7t + 18t^2).
If W = 1/5(X − 2) what are the expected value and variance of W?

Solution:

Moment generating function of $X \sim N(\mu,\sigma^2)$ is $M_X(t) = e^{t\mu+\frac{1}{2}\sigma^2 t^2}$

Since $M_X(t) = e^{-7t+18t^2} = e^{t(-7)+\frac{1}{2}(36)t^2}$ then $X \sim N(-7,6^2)$

So E(X) = -7 and Var(X) = 36

E(W) = E(1/5(X-2)) = (1/5)E(X-2) = (1/5)(E(X)-E(2)) = (1/5)(-7-2) = -9/5 = -1.8

Var(W) = Var(1/5(X-2)) = (1/25)Var(X-2) = (1/25)Var(X) = 36/25 = 1.44

Answer: E(W) = -1.8 ,Var(W) = 1.44

10. (a)
The random variable X has moment generating function
mx(t)=(1−2t)^−15/2 .
If Z = 1/3(X + 3), what is the variance V(Z)?

Solution:

Moment generating function of $X \sim \lambda^2_k$ is $M_X(t) = \frac{1}{(1-2t)^{k/2}}$, E(X) =k , Var(X) = 2k

Since $M_X(t) = \frac{1}{(1-2t)^{15/2}}$ then $X \sim \lambda^2_{15}$, E(X) = 15 , Var(X) = 30

Var(Z) = Var(1/3(X+3)) = (1/9)Var(X+3) = (1/9)Var(X) = 30/9 = 10/3

Answer: Var(Z) = 10/3

(b) Y is a random variable with moment generating function mY (t) = (0.8 + 0.2e^t)^10
.
What is $P(1 \le Y \le 4)$?

Solution:

Moment generating function of X~Bin(n.p) is $M_X(t) = (1 - p + pe^t)^n$

Since $M_Y(t) = (1 - 0.2 + 0.2e^t)^{10}$ then Y~Bin(10,0.2)

P(1≤Y≤4) = P(Y=1)+P(Y=2)+P(Y=3)+P(Y=4)

$P(Y=1) = {}_{10}C_1(0.2)^1(0.8)^9 = 0.2684$

$P(Y=2) = {}_{10}C_2(0.2)^2(0.8)^8 = 0.302$

$P(Y=3) = {}_{10}C_3(0.2)^3(0.8)^7 = 0.2013$

$P(Y=4) = {}_{10}C_4(0.2)^4(0.8)^6 = 0.0881$

P(1≤Y≤4) = 0.8598

Answer: 0.8598

11. (a)
The random variable X takes the values 0, 1, 2 with probabilities 1/2 , 3/8, 1/8 respectively. Find the moment generating function for X and verify that the second moment of X is 7/8.
[5]

Solution:

$$M_X(t) = E(e^{tX}) = e^{t(0)}\left(\frac{1}{2}\right) + e^{t(1)}\left(\frac{3}{8}\right) + e^{t(2)}\left(\frac{1}{8}\right) = \frac{1}{2} + \frac{3}{8}e^t + \frac{1}{8}e^{2t}$$

$$M_X(t) = \frac{1}{2} + \frac{3}{8}e^t + \frac{1}{8}e^{2t}$$

$$M'_X(t) = \frac{3}{8}e^t + \frac{1}{4}e^{2t}$$

$$M''_X(t) = \frac{3}{8}e^t + \frac{1}{2}e^{2t}$$

$$M''_X(0) = \frac{3}{8}e^0 + \frac{1}{2}e^{2(0)} = \frac{3}{8} + \frac{1}{2} = \frac{7}{8}$$

Second moment of X is: $E(X^2) = M''_X(0) = 7/8$

(b)
The random variable Z has distribution function FZ (z) = 1 − exp(−5z) for z ∈ [0, ∞).

Find the moment generating function for Z and verify that the first moment of Z is 0.2.

[5]

Solution:

Since $F_Z(z) = 1 - e^{-5z}$ for $z \in [0, \infty)$

We have : $f_Z(z) = F'_Z(z) = 5e^{-5z}$ for $z \in [0, \infty)$

$$M_Z(t) = E(e^{tZ}) = \int_0^{\infty} e^{tz}(5e^{-5z})dz = 5\int_0^{\infty} e^{(t-5)z}dz$$

If $t < 5$ we have:

$$5\int_0^{\infty} e^{(t-5)z}dz = 5\left(\frac{e^{(t-5)z}}{t-5}\right)_0^{\infty} = 5\left(0 - \left(\frac{1}{t-5}\right)\right) = \frac{5}{5-t}$$

Then: $M_Z(t) = \frac{5}{5-t}$

First moment of Z is $E(Z) = M'_Z(0)$

Since $M_Z(t) = \frac{5}{5-t}$ we have: $M'_Z(t) = \frac{5}{(5-t)^2} \rightarrow M'_Z(0) = \frac{5}{(5-0)^2} = \frac{1}{5} = 0.2$

Then $E(Z) = 0.2$